Impedance Matching with Transmission Lines

Introduction
Transmission lines have some special properties that can make them advantageous for impedance matching. First they can provide an open at certain frequencies, and a short at others, making them useful for biasing, where an impedance needs to be a short at DC, but an open at a desired frequency. This property also allows them to short out harmonics of a signal, without attenuation of the desired signal. At very high frequencies (>2GHz), the physical dimensions of inductors and capacitor make them difficult to manufacture. At these frequencies, inductors and capacitors are replaced with transmission line, made cheap and small out of PC board.

Inputs

\[ Z_L := 100\Omega \]
\[ Z_S := 50\Omega \]
\[ S_{11\text{max}} := -10\text{dB} \]
\[ f_c := 1\text{GHz} \]
\[ \Delta f := 0.3\text{GHz} \]
\[ \varepsilon_r := 4.2 \]
\[ h := 60\text{mil} \]
Transmission Line Synthesis Function

\[
W(e_r, h, Z_0) := \begin{cases} 
A & \frac{Z_0}{60\text{ohm}} \sqrt{\frac{e_r + 1}{2} + \frac{e_r - 1}{e_r + 1} \left( \frac{0.23 + 0.11}{e_r} \right)} \\
B & \frac{377\text{ohm}\pi}{2\cdot Z_0\sqrt{e_r}} \\
W_1 & \frac{8\cdot e^A\cdot h}{e^2 \cdot A - 2} \\
W_2 & \frac{2\cdot h}{\pi} \left[ B - 1 - \ln(2\cdot B - 1) + \frac{e_r - 1}{2\cdot e_r} \left( \ln(B - 1) + 0.39 - \frac{0.61}{e_r} \right) \right] \\
W & \text{if} \left( \frac{W_1}{h} < 2, \text{if} \left( \frac{W_2}{h} > 2, W_2, 0 \right) \right)
\end{cases}
\]

Single Shunt Stub Transmission Line Matching

Here a stub and a transmission line are used to match a transmission line input, with characteristic impedance of \(Z_S\), to a complex load of impedance \(Z_L\). A open circuit stub, of length \(l\), is placed at the input with a transmission line connecting it to the load with a length, \(d\). Both transmission line segments have a characteristic impedance of \(Z_S\). You little (choice of two implementations) control over the bandwidth for single stub tuners.

\[
W_{\text{val}} := W(e_r, h, Z_S)
\]

\[
\omega_c := 2\cdot \pi \cdot f_c
\]

\[
\lambda := \frac{2\cdot \pi \cdot c}{\omega_c}
\]

\[
R_L := \text{Re}(Z_L)
\]

\[
X_L := \text{Im}(Z_L)
\]

\[
W_{\text{val}} = 3.016\text{mm}
\]

\[
\omega_c = 6.283 \times 10^9 \frac{\text{rad}}{\text{sec}}
\]

\[
\lambda = 29.9\text{cm}
\]

\[
R_L = 100\Omega
\]

\[
X_L = 0\Omega
\]

When designing transmission line transformers, there are multiple solutions to the matching problem. For an example, an additional wavelength of transmission line can be added to yield the same matching, but with a different frequency response. This is usually not done to keep the area small. Here multiple solutions are presented.

\[
t_1 := \frac{X_L + \frac{R_L}{Z_S} \left( [Z_S - R_L]^2 + X_L^2 \right)}{R_L - Z_S}
\]

\[
t_2 := \frac{X_L - \frac{R_L}{Z_S} \left( [Z_S - R_L]^2 + X_L^2 \right)}{R_L - Z_S}
\]

\[
t_1 = 1.414
\]

\[
t_2 = -1.414
\]
Is Solution #1 OK to use?

\[ \text{Area of Solution #1} = 0.488 \text{cm}^2 \]

Is Solution #2 OK to use?

\[ \text{Area of Solution #2} = 4.021 \text{cm}^2 \]

Is Solution #3 OK to use?

\[ \text{Area of Solution #3} = 2.742 \text{cm}^2 \]

Is Solution #4 OK to use?

\[ \text{Area of Solution #4} = 1.767 \text{cm}^2 \]

Some solutions are not valid (negative lengths), so they are dropped:

Is Solution #1 OK to use?

\[ d_1 = 4.546 \text{ cm} \]

Is Solution #2 OK to use?

\[ d_2 = 10.404 \text{ cm} \]

Connecting Transmission Line (Solution #1&3)

\[ \frac{1}{B_{s1}} = -70.711 \text{ ohm} \]

Susceptance of Stub (Solution #1)

\[ l_{o1} = -29.289 \text{ mm} \]

Length of Open Circuit Stut (Solution #1)

\[ l_{o2} = 45.461 \text{ mm} \]

Length of Short Circuit Stut (Solution #3)

\[ l_{o2} = -29.289 \text{ mm} \]

Length of Short Circuit Stut (Solution #4)

\[ l_{o2} = -45.461 \text{ mm} \]

Is Solution #3 OK to use?

\[ l_{o1} = 29.289 \text{ mm} \]

Is Solution #4 OK to use?

\[ l_{o1} = -29.289 \text{ mm} \]

Area for Solution #1

\[ \text{Area} = 0.488 \text{ cm}^2 \]

Area for Solution #2

\[ \text{Area} = 4.021 \text{ cm}^2 \]

Area for Solution #1

\[ \text{Area} = 2.742 \text{ cm}^2 \]

Area for Solution #4

\[ \text{Area} = 1.767 \text{ cm}^2 \]

Areas if white space is considered

\[ l_{o1} \cdot d_1 = -13.315 \text{ cm}^2 \]

\[ l_{o2} \cdot d_2 = 30.472 \text{ cm}^2 \]

\[ l_{s1} \cdot d_1 = 20.667 \text{ cm}^2 \]

\[ l_{s2} \cdot d_2 = -47.297 \text{ cm}^2 \]
Single Series Stub Transmission Line Matching

\[
G_L := \text{Re} \left( \frac{1}{Z_L} \right) \quad B_L := \text{Im} \left( \frac{1}{Z_L} \right)
\]

\[
Y_S := \frac{1}{Z_S}
\]

\[
B_L + \frac{G_L}{\sqrt{Y_S}} \left[ (Y_S - G_L)^2 + B_L^2 \right]^{1/2}
\]

\[
t_1 := \frac{B_L - \frac{G_L}{\sqrt{Y_S}} \left[ (Y_S - G_L)^2 + B_L^2 \right]^{1/2}}{\sqrt{G_L - Y_S}}
\]

\[
t_2 := \frac{B_L - \frac{G_L}{\sqrt{Y_S}} \left[ (Y_S - G_L)^2 + B_L^2 \right]^{1/2}}{\sqrt{G_L - Y_S}}
\]

\[
d_1 := \frac{\lambda}{2\pi} \cdot \text{atan}(t_1)
\]

\[
d_2 := \frac{\lambda}{2\pi} \left( \pi + \text{atan}(t_2) \right)
\]

\[
X_{s1} := \frac{G_L \cdot t_1 - \left( Y_S - Y_S \cdot B_L \cdot B_L \cdot t_1 \cdot Y_S \right)}{-Y_S \left[ G_L^2 + \left( B_L + t_1 \cdot Y_S \right)^2 \right]^{1/2}}
\]

\[
X_{s2} := \frac{G_L \cdot t_2 - \left( Y_S - Y_S \cdot B_L \cdot B_L \cdot t_2 \cdot Y_S \right)}{-Y_S \left[ G_L^2 + \left( B_L + t_2 \cdot Y_S \right)^2 \right]^{1/2}}
\]

\[
l_{o1} := \frac{\lambda}{2\pi} \cdot \text{atan} \left( \frac{Z_S}{X_{s1}} \right)
\]

\[
l_{s1} := \frac{\lambda}{2\pi} \cdot \text{atan} \left( \frac{X_{s1}}{Z_S} \right)
\]

\[
l_{o2} := \frac{\lambda}{2\pi} \cdot \text{atan} \left( \frac{Z_S}{X_{s2}} \right)
\]

\[
l_{s2} := \frac{\lambda}{2\pi} \cdot \text{atan} \left( \frac{X_{s2}}{Z_S} \right)
\]

\[
\text{OK}_1 := \left( l_{o1} > 0 \right) \cdot \left( d_1 > 0 \right)
\]

\[
\text{OK}_2 := \left( l_{o2} > 0 \right) \cdot \left( d_2 > 0 \right)
\]

\[
\text{OK}_3 := \left( l_{s1} > 0 \right) \cdot \left( d_1 > 0 \right)
\]

\[
\text{OK}_4 := \left( l_{s2} > 0 \right) \cdot \left( d_2 > 0 \right)
\]

\[
\text{Area}_1 := \left( l_{o1} + d_1 \right) \cdot W_{\text{val}}
\]

\[
\text{Area}_2 := \left( l_{o2} + d_2 \right) \cdot W_{\text{val}}
\]

\[
\text{Area}_3 := \left( l_{s1} + d_1 \right) \cdot W_{\text{val}}
\]

\[
\text{Area}_4 := \left( l_{s2} + d_2 \right) \cdot W_{\text{val}}
\]

---

Fig. 1: Single series stub matching

Load Conductance and Susceptance

Source Admittance

Coefficient for First Solution

Coefficient for Second Solution

<table>
<thead>
<tr>
<th>Solution</th>
<th>Length of Open Circuit Stub</th>
<th>Length of Short Circuit Stub</th>
<th>Is Solution OK to use?</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>45.461 mm</td>
<td>0.488 cm²</td>
<td>OK_1 = 0</td>
</tr>
<tr>
<td>#2</td>
<td>29.289 mm</td>
<td>4.021 cm²</td>
<td>OK_2 = 0</td>
</tr>
<tr>
<td>#3</td>
<td>29.289 mm</td>
<td>-1.767 cm²</td>
<td>OK_3 = 0</td>
</tr>
<tr>
<td>#4</td>
<td>45.461 mm</td>
<td>6.276 cm²</td>
<td>OK_4 = 1</td>
</tr>
</tbody>
</table>

Area for Solution

<table>
<thead>
<tr>
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<th>Area</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.488 cm²</td>
<td>Area for Solution #1</td>
</tr>
<tr>
<td>#2</td>
<td>4.021 cm²</td>
<td>Area for Solution #2</td>
</tr>
<tr>
<td>#3</td>
<td>-1.767 cm²</td>
<td>Area for Solution #1</td>
</tr>
<tr>
<td>#4</td>
<td>6.276 cm²</td>
<td>Area for Solution #2</td>
</tr>
</tbody>
</table>
1/4 Wave Transformer Matching

Fig. 1: Single shunt stub matching

No control over bandwidth achieved

\[ \Delta f_{f_c} := \frac{\Delta f}{f_c} \]
\[ S_{11\text{max}} \]
\[ \Gamma_{\text{max}} := 10 \]
\[ Z_{1/4} := \sqrt{Z_S/Z_L} \]
\[ \Delta f_{f_c\text{max}} := 2 - \frac{4}{\pi} \cdot \cos \left( \frac{\Gamma_{\text{max}}}{2} \frac{2\sqrt{Z_S/Z_L}}{|Z_L - Z_S|} \right) \]

OK := \Delta f_{f_c\text{max}} > \Delta f_{f_c} \]
\[ \beta := \frac{2 \pi}{\lambda} \sqrt{\varepsilon_r} \]
\[ \text{len} := \frac{\lambda}{4} \]
\[ W_{\text{val}} := W(\varepsilon_r, h, Z_{1/4}) \]
\[ \text{Area} := \text{len} \cdot W_{\text{val}} \]

\textbf{Outputs}

\[ Z_{1/4} = 70.711 \Omega \]
OK = 1
Area = 1.203 cm\(^2\)

\[ \Delta f_{f_c} = 30\% \]
\[ \Gamma_{\text{max}} = 0.1 \]
\[ Z_{1/4} = 70.711 \Omega \]
\[ \Delta f_{f_c\text{max}} = 36.7\% \]
\[ \beta = 43.066 \text{ m}^{-1} \]
\[ \text{len} = 7.475 \text{ cm} \]
\[ W_{\text{val}} = 1.61 \text{ mm} \]
\[ \text{Area} = 1.203 \text{ cm}^2 \]

\[ \text{OK to use this circuit?} \]

\textbf{Fractional Bandwidth Needed}
\textbf{Reflection Coefficient Needed}
\textbf{Impedance of Quarter Wave Transm}
\textbf{Actual Bandwidth}
\textbf{OK to use this circuit?}
\textbf{Phase Constant (Wave Number)}
\textbf{Length of Transformer}
\textbf{Width of Transformer}
\textbf{Area of Transformer}
\textbf{Impedance of Line}
\textbf{Area of Transformer}
Reflection Coefficients
\[ \Gamma_i^N, () = \Gamma_{\text{max}}^N, () A^N, () C^N, () \cdot := \binomial{N}{i} \]

Binomial Coefficients
\[ \binomial{N}{i} = \frac{N!}{(N-i)! \cdot i!} \]

Index Vector
\[ i := 0..(N-1) \]
\[ C(n, N) := \frac{N!}{(N-n)! \cdot n!} \]
\[ C(i, N) = \begin{cases} 1 & \text{if } i = 0 \\ 0.167 & \text{otherwise} \end{cases} \]

Reflection Coefficients
\[ \Gamma(n, N) := A(N) \cdot C(n, N) \]
\[ \Gamma(i, N) = \begin{cases} 0.167 & \text{if } i = 0 \\ \text{NaN} & \text{otherwise} \end{cases} \]

DC Reflection Coefficient
\[ \Gamma_{\text{DC}} = 0.333 \]

Fractional Order Needed
\[ \ln \left( \frac{\Gamma_{\text{max}}}{\Gamma_{\text{DC}}} \right) = 0.828 \]

Order Estimation
\[ N := \text{ceil} \left( \frac{\ln \left( \frac{\Gamma_{\text{max}}}{\Gamma_{\text{DC}}} \right)}{\ln \left( \cos \left( \frac{\pi}{2} \left( 1 - \frac{\Delta f_c}{2} \right) \right) \right)} \right) \]
\[ N = 1 \]

Binomial Coefficient
\[ A(N) = 0.167 \]

Actual Fractional Bandwidth
\[ \Delta f_c(N) = 38.795\% \]

Fig. 1: 1/4 wave binomial transformer matching

\[ \Gamma_{\text{DC}} := \frac{Z_L - Z_S}{Z_L + Z_S} \]

\[ \ln \left( \frac{\Gamma_{\text{max}}}{\Gamma_{\text{DC}}} \right) = 0.828 \]

\[ \frac{\ln \left( \frac{\Gamma_{\text{max}}}{\Gamma_{\text{DC}}} \right)}{\ln \left( \cos \left( \frac{\pi}{2} \left( 1 - \frac{\Delta f_c}{2} \right) \right) \right)} \]

\[ \text{Index Vector} \]
\[ C(i, N) \]

\[ \text{Reflection Coefficients} \]
\[ \Gamma(i, N) \]

\[ Z_L, Z_S, \lambda, \lambda/4, N = \]
j := 1.. N
Z := [Zval_0 ← Z_S  
for i ∈ 0.. (N - 1)  
Zval_{i+1} ← Zval_i \left( \frac{\Gamma(i, N) + 1}{1 - \Gamma(i, N)} \right)  
Zval]

\text{Wval}_j := W(\varepsilon_r, h, Z_j)

\text{max}(\text{Wval}) = 1.643 \, \text{mm}

\text{len} := \frac{\lambda}{4} \cdot N

\text{Area} := \text{len} \cdot \text{max}(\text{Wval})

\textbf{Outputs}

\text{Wval} = \left( \begin{array}{c} 0 \\ 1.643 \end{array} \right) \, \text{mm}

\text{len} = 7.475 \, \text{cm}

\text{Area} = 1.228 \, \text{cm}^2

\textbf{T-Line Impedances}

Z_j = \frac{\text{70}}{70} \, \Omega

\textbf{Widths of Transmission Line Segments}

\text{Wval} = \left( \begin{array}{c} 0 \\ 1.643 \end{array} \right) \, \text{mm}

\text{Maximum Width}

\text{len} = 7.475 \, \text{cm}

\text{Area} = 1.228 \, \text{cm}^2

\textbf{Widths of Line Segments}

\textbf{Length of Transformer}

\textbf{Area of Transformer}
1/4 Wave Chebyshev Transformer Matching

\[ \theta_{\text{max}} := \text{asec} \left( \text{cosh} \left( \frac{1}{N} \text{acosh} \left( \frac{\ln \left( \frac{Z_L}{Z_S} \right)}{2 \Gamma_{\text{max}}} \right) \right) \right) = 47.993 \text{deg} \]

\[ \Delta f_{c}(N) := 2 - \frac{4}{\pi} \cdot \text{asec} \left( \frac{1}{N} \cdot \text{acosh} \left( \frac{\Gamma_{\text{DC}}}{\Gamma_{\text{max}}} \right) \right) \]

\[ \Delta f_{c}(N) = 95.098\% \]

\[ \Lambda := \Gamma_{\text{max}} \]

\[ \Lambda = 0.1 \]

Fractional Order Needed

Order Estimation

Actual Fractional Bandwidth

Fig. 1: 1/4 wave chebyshev transformer matching

Transformer Coefficient
\[ T(n, x) := \text{if}(x < 1, \cos(n - \cos(x)), \cosh(n - \cosh(x))) \]
\[ i := 0..N \]
\[ \Gamma(n) := \begin{cases} \frac{A}{2} \cdot \sec(\theta_{\text{max}})^3 & \text{if } n = 0 \\ \frac{3}{2} \cdot A \cdot \left( \sec(\theta_{\text{max}})^3 - \sec(\theta_{\text{max}}) \right) & \text{if } n = 1 \\ \frac{3}{2} \cdot A \cdot \left( \sec(\theta_{\text{max}})^3 - \sec(\theta_{\text{max}}) \right) & \text{if } n = 2 \\ \frac{A}{2} \cdot \sec(\theta_{\text{max}})^3 & \text{if } n = 3 \end{cases} \]
\[ \Gamma(i) = \begin{bmatrix} 0.167 \\ 0.276 \\ 0.276 \end{bmatrix} \]
\[ j := 1..N \]
\[ Z := \begin{cases} Z_{\text{val}, 0} \leftarrow Z_S \\ \text{for } i \in 0..N - 1 \\ Z_{\text{val}, i+1} \leftarrow Z_{\text{val}, i} e^{2 \cdot \Gamma(i)} \\ Z_{\text{val}} \end{cases} \]
\[ W_{\text{val}, j} := W(\varepsilon_t, h, Z_j) \]
\[ \text{max}(W_{\text{val}}) = 1.652 \text{ mm} \]
\[ \text{len} := \frac{\lambda}{4} \cdot N \]
\[ \text{Area} := \text{len} \cdot \text{max}(W_{\text{val}}) \]
\[ \text{Transmission Line Impedances} \]
\[ Z_j = \begin{bmatrix} 69.803 \\ 121.31 \end{bmatrix} \Omega \]
\[ \text{Widths of Transmission Line Segments} \]
\[ W_{\text{val}} = \begin{bmatrix} 0 \\ 1.652 \\ 0.401 \end{bmatrix} \text{ mm} \]
\[ \text{Maximum Width} \]
\[ \text{len} = 14.95 \text{ cm} \]
\[ \text{Length of Transformer} \]
\[ \text{Area} = 2.47 \text{ cm}^2 \]
\[ \text{Area of Transformer} \]
**Exponential Taper Matching Transformer**

\[
L := \frac{\pi}{\beta} \\
L_2 := \frac{2\pi}{\beta} \\
Z(z) := Z_S e^{\frac{1}{L} \ln \left( \frac{Z_L}{Z_S} \right) z} \\
Z_2(z) := Z_S e^{\frac{1}{L_2} \ln \left( \frac{Z_L}{Z_S} \right) z}
\]

- **Length of Transformer (Option #1: \(\beta L = \pi\))**
  \[L_1 = 7.295\text{ cm}\]
  \[L_2 = 14.59\text{ cm}\]

- **Length of Transformer (Option #2: \(\beta L = 2\pi\))**
  \[L_2 = 2\pi\cdot \beta\]

\[
W(e_r, h, Z(0\text{ cm})) = 3.016\text{ mm} \\
W(e_r, h, Z(L)) = 0.714\text{ mm} \\
\text{TotArea} := L \cdot W(e_r, h, Z(0\text{ cm}))
\]

- **Width at Beginning of Taper**
- **Width at End of Taper**
- **Total Area of Transformer**

\[\text{TotArea} = 2.2\text{ cm}^2\]
Triangular Taper Matching Transformer

\[ L := \frac{2 \pi \beta}{\alpha} \]

\[ Z(z) := \begin{cases} 
\frac{1}{2} \cdot Z_S e^{2 \left( \frac{z}{L} \right)^2 \cdot \ln \left( \frac{Z_L}{Z_S} \right)} & \text{if } z < \frac{L}{2} \cdot Z_S e^{4 \left( \frac{z}{L} \right)^2 \cdot \ln \left( \frac{Z_L}{Z_S} \right)}
\end{cases} \]

Length of Transformer (Option #1)

\[ L = 14.59 \text{ cm} \]

Width at Beginning of Taper

\[ W(e_r, h, Z(0 \text{ cm})) = 3.016 \text{ mm} \]

Width at End of Taper

\[ W(e_r, h, Z(L)) = 0.714 \text{ mm} \]

Total Area of Transformer

\[ \text{TotArea} := L \cdot W(e_r, h, Z(0 \text{ cm})) \]

\[ \text{TotArea} = 4.401 \text{ cm}^2 \]
Klopfenstein Taper Matching Transformer

Fig. 1: Klopfenstein transformer matching

\[
A := \text{acosh} \left( \frac{\Gamma_{DC}}{\Gamma_{\text{max}}} \right)
\]

\[
L := \frac{A}{\beta}
\]

\[
\phi(x, A) := \int_{0}^{x} \frac{\Pi(A - 1 - y^2)}{A\sqrt{1 - y^2}} \, dy
\]

\[
Z(z) := Z_S e^{-\frac{1}{2} \ln \left( \frac{Z_L}{Z_S} \right) - \frac{\Gamma_{DC}}{\cos h(A)} A^2 \phi \left( \frac{2z}{L} - 1, A \right)}
\]

\[
\Gamma(\beta L) := \Gamma_{DC} e^{-\beta L \cos \left( \frac{\sqrt{\beta L^2 - A^2}}{\cosh(A)} \right)}
\]

\[
W(\varepsilon_r, h, Z(0\text{cm})) = 2.492 \text{ mm}
\]

\[
W(\varepsilon_r, h, Z(L)) = 0.956 \text{ mm}
\]

\[
\text{TotArea} := L \cdot W(\varepsilon_r, h, Z(0\text{cm}))
\]

\[
\text{TotArea} = 1.084 \text{ cm}^2
\]

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