

Butterworth LC Filter Designer

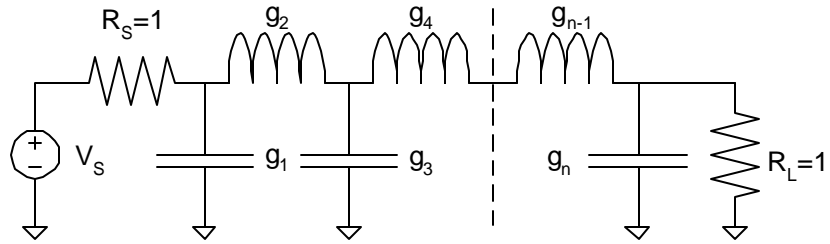


Fig. 1: LC filter used for odd-order analysis

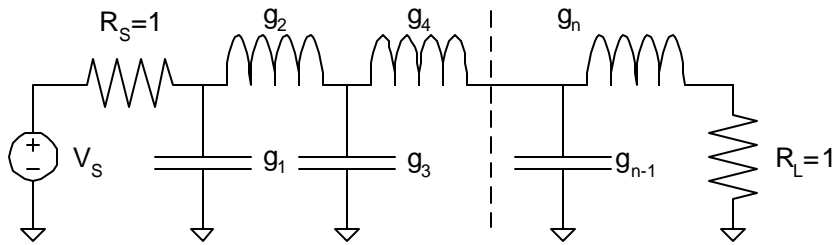


Fig. 2: Filter used for even order analysis

- ▶ useful functions and identities
- ▶ Units
- ▶ Constants

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Introduction

This routine calculates the required inductances and capacitances for a lowpass LC butterworth filter. It is special in that it also sizes the components to deliver maximum power from a given source resistance to a different load resistance. Reactive portions of the source and load must be removed subtracted the filter network, so it is usually desirable to choose an odd order network for maximum power transfer. Given the lowpass LC coefficients, the network can easily be transformed into a bandpass, highpass, notch filter based on other topologies: gm/C, active RC, etc. A butterworth filter has the following power transfer function, where the order of the filter is given by n:

$$A = 10 \cdot \log \left[1 + \left(\frac{f}{f_c} \right)^{2 \cdot n} \right]$$

Inputs

Atten := 80dB
 Ripple := 1dB
 f_p := 1kHz
 f_s := 10kHz
 R_S := 10k Ω
 R_L := 20k Ω

Stop-Band Attenuation
Maximum Passband Ripple
Passband corner frequency
Stopband corner frequency
Source Impedance
Load Impedance

Order and -3dB Corner Frequency Estimation

$$n = \text{ceil} \left(\frac{\ln \left(\frac{\text{Atten}}{10^{10} - 1} \right)}{2 \cdot \ln \left(\frac{f_s}{f_c} \right)} \right)$$

Required Order of Butterworth Filter for 3dB droop

Function 1: Holds Atten Constant and chooses n to meet Ripple

```

nf_c := | n ← 1
        | f_c ←  $\frac{f_s}{\left( \frac{\text{Atten}}{10^{10} - 1} \right)^{\frac{1}{2 \cdot n}}}$ 
        | Aatfp ←  $10 \cdot \log \left[ 1 + \left( \frac{f_p}{f_c} \right)^{2 \cdot n} \right]$ 
        | Aatfs ←  $10 \cdot \log \left[ 1 + \left( \frac{f_s}{f_c} \right)^{2 \cdot n} \right]$ 
        | while (Aatfs ≤ Atten) · (Aatfp > Ripple)
        |   | n ← n + 1
        |   | f_c ←  $\frac{f_s}{\left( \frac{\text{Atten}}{10^{10} - 1} \right)^{\frac{1}{2 \cdot n}}}$ 
        |   | Aatfp ←  $10 \cdot \log \left[ 1 + \left( \frac{f_p}{f_c} \right)^{2 \cdot n} \right]$ 
        |   | Aatfs ←  $10 \cdot \log \left[ 1 + \left( \frac{f_s}{f_c} \right)^{2 \cdot n} \right]$ 
        |   |  $\left( \frac{n}{\frac{f_c}{\text{Hz}}} \right)$ 
n := nf_c_1

```

n = 5

Number of Poles

f_c := nf_c_2 · Hz

f_c = 1.585 kHz

3 dB Corner Frequency

Aatfp := $10 \cdot \log \left[1 + \left(\frac{f_p}{f_c} \right)^{2 \cdot n} \right]$

Aatfp = 0.043 dB

Passband Ripple

Aatfs := $10 \cdot \log \left[1 + \left(\frac{f_s}{f_c} \right)^{2 \cdot n} \right]$

Aatfs = 80 dB

Stopband Attenuation

Function 2: Holds Ripple Constant and chooses n to meet Atten

```

nf_c :=
  n ← 1
  f_c ←  $\frac{f_p}{\left(\frac{\text{Ripple}}{10^{10} - 1}\right)^{\frac{1}{2 \cdot n}}}$ 
  Aatfp ←  $10 \cdot \log \left[ 1 + \left(\frac{f_p}{f_c}\right)^{2 \cdot n} \right]$ 
  Aatfs ←  $10 \cdot \log \left[ 1 + \left(\frac{f_s}{f_c}\right)^{2 \cdot n} \right]$ 
  while (Aatfs ≤ Atten) · (Aatfp ≥ Ripple)
    n ← n + 1
    f_c ←  $\frac{f_p}{\left(\frac{\text{Ripple}}{10^{10} - 1}\right)^{\frac{1}{2 \cdot n}}}$ 
    Aatfp ←  $10 \cdot \log \left[ 1 + \left(\frac{f_p}{f_c}\right)^{2 \cdot n} \right]$ 
    Aatfs ←  $10 \cdot \log \left[ 1 + \left(\frac{f_s}{f_c}\right)^{2 \cdot n} \right]$ 
   $\left( \begin{array}{l} n \\ f_c \\ \text{Hz} \end{array} \right)$ 

```

$n := nf_{c_1}$

$n = 5$

Number of Poles

$f_c := nf_{c_2} \cdot \text{Hz}$

$f_c = 1.145 \text{ kHz}$

3 dB Corner Frequency

$Aatfp := 10 \cdot \log \left[1 + \left(\frac{f_p}{f_c}\right)^{2 \cdot n} \right]$

$Aatfp = 1 \text{ dB}$

Passband Ripple

$Aatfs := 10 \cdot \log \left[1 + \left(\frac{f_s}{f_c}\right)^{2 \cdot n} \right]$

$Aatfs = 94.132 \text{ dB}$

Stopband Attenuation

Poles and Zeros

$$k := 1..n$$

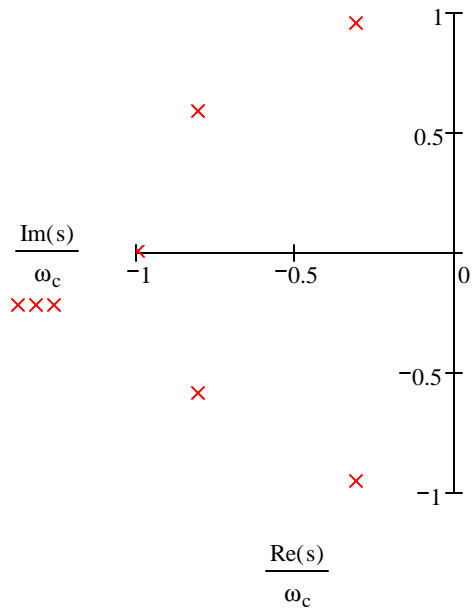
Index Vector for poles

$$\omega_c := 2 \cdot \pi \cdot f_c$$

$$s_k := \omega_c \cdot e^{\frac{j \cdot (2 \cdot k + n - 1) \cdot \pi}{2 \cdot n}}$$

Poles

$$s^T = (-2.223 + 6.84i \quad -5.819 + 4.227i \quad -7.192 \quad -5.819 - 4.227i \quad -2.223 - 6.84i) \text{ kHz}$$



Quadratic Sections

$$\frac{1}{\frac{s^2}{re^2 + im^2} + \left(\frac{2 \cdot re}{re^2 + im^2}\right)s + 1} = \frac{1}{\left(\frac{s}{\omega}\right)^2 + \frac{1}{Q} \cdot \frac{s}{\omega} + 1} = \frac{1}{\left[\left(\frac{s}{\omega}\right)^2 + 2 \cdot \zeta \cdot \frac{s}{\omega} + 1\right]}$$

$$m := 1.. \text{floor}\left(\frac{n}{2}\right)$$

$$\omega_m := \sqrt{\text{Re}(s_m)^2 + \text{Im}(s_m)^2}$$

$$\frac{\omega}{2 \cdot \pi} = (1.145 \quad 1.145) \text{ kHz}$$

Quadratic center frequenc
butterworth)

$$Q_m := \frac{-\omega_m}{2 \cdot \text{Re}(s_m)}$$

$$Q^T = (1.618 \quad 0.618)$$

Quadratic Q

$$\zeta_m := \frac{-\text{Re}(s_m)}{\omega_m}$$

$$\zeta^T = (0.309 \quad 0.809)$$

Quadratic Damping Fact

$$\omega_{\text{ceil}\left(\frac{n}{2}\right)} := \text{if}\left(\text{ceil}\left(\frac{n}{2}\right) > \frac{n}{2}, -s_{\text{ceil}\left(\frac{n}{2}\right)}, 0 \frac{\text{rad}}{\text{sec}}\right)$$

$$\frac{\omega}{2 \cdot \pi} = (1.145 \quad 1.145 \quad 1.145) \text{ kHz} \quad \text{First Order Section}$$

Transfer Function

$$\alpha_k := \prod_{m=1}^k \frac{\cos\left[\frac{(m-1)\cdot\pi}{2\cdot n}\right]}{\sin\left(\frac{m\cdot\pi}{2\cdot n}\right)}$$

$$\alpha^T = (3.236 \ 5.236 \ 5.236 \ 3.236 \ 1)$$

Transfer Function Coefficients
(α_n always equals 1)

$$M(s) := \frac{1}{1 + \sum_{k=1}^n \alpha_k \cdot \left(\frac{s}{\omega_c}\right)^k}$$

Transfer Function

$$gd(f) := \frac{d}{df} \arg \left[\frac{1}{1 + \sum_{k=1}^n \alpha_k \cdot \left(\frac{j\cdot f}{f_c}\right)^k} \right]$$

Group Delay Function

$$\text{phase}(p) := \begin{cases} \longrightarrow \\ p \leftarrow \arg(p) \\ \text{return } p \text{ if } \text{last}(p) < 2 \\ \text{wrap} \leftarrow 0 \\ \text{for } i \in 2.. \text{last}(p) \\ \left| \begin{array}{l} \text{wrap} \leftarrow \text{wrap} - 2\cdot\pi \text{ if } (p_i - p_{i-1} + \text{wrap}) > 3 \\ \text{wrap} \leftarrow \text{wrap} + 2\cdot\pi \text{ if } (p_i - p_{i-1} + \text{wrap}) < -3 \\ p_i \leftarrow p_i + \text{wrap} \end{array} \right. \\ p \end{cases}$$

Phase Function

$$M_2(f) := \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2\cdot n}}}$$

Magnitude Transfer Function
for 3dB droop

$$\left(|S_{21}(s)|\right)^2 = \frac{K_n}{1 + \varepsilon^2 \cdot \left(\frac{f}{f_c}\right)^{2\cdot n}}$$

$$\alpha_{\max} = 10 \cdot \log(1 + \varepsilon^2)$$

Power Transfer Function for α_{\max} dB droop

$$\varepsilon = \sqrt{10^{\frac{\alpha_{\max}}{10}} - 1}$$

$$M_3(s) := \frac{1}{\sqrt{1 + (-1)^n \cdot \left(\frac{s}{\omega_c}\right)^{2\cdot n}}}$$

Transfer Function (Eq. 2.10b)

Plotting

$\text{num} := 100$

$i := 1.. \text{num}$

$f_{\text{start}} := \frac{f_p}{5}$

$f_{\text{stop}} := f_s \cdot 2$

$f_i := \left(\frac{f_{\text{stop}}}{f_{\text{start}}} \right)^{\frac{i-1}{\text{num}-1}} \cdot f_{\text{start}}$

$\omega_i := 2 \cdot \pi \cdot f_i$

$s_i := j \cdot \omega_i$

$\text{ang} := \text{phase}(\overrightarrow{M(s)})$

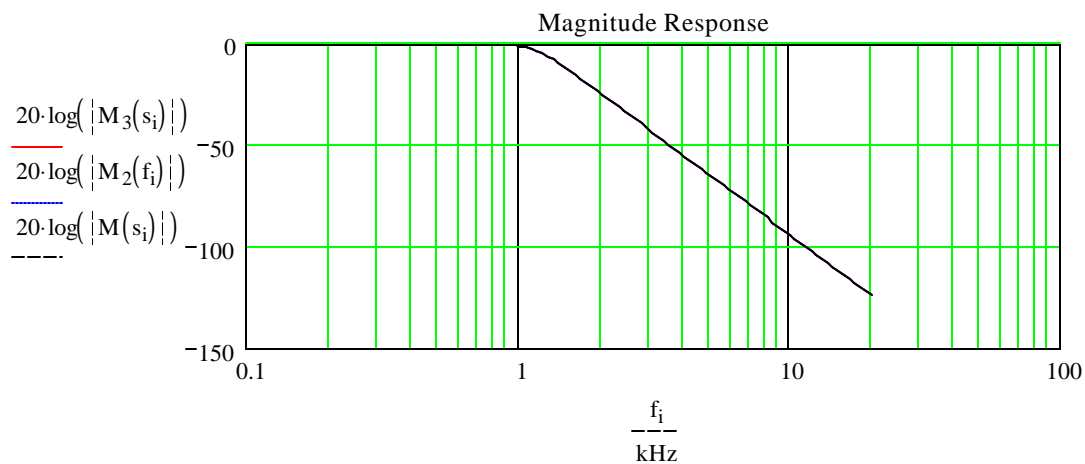
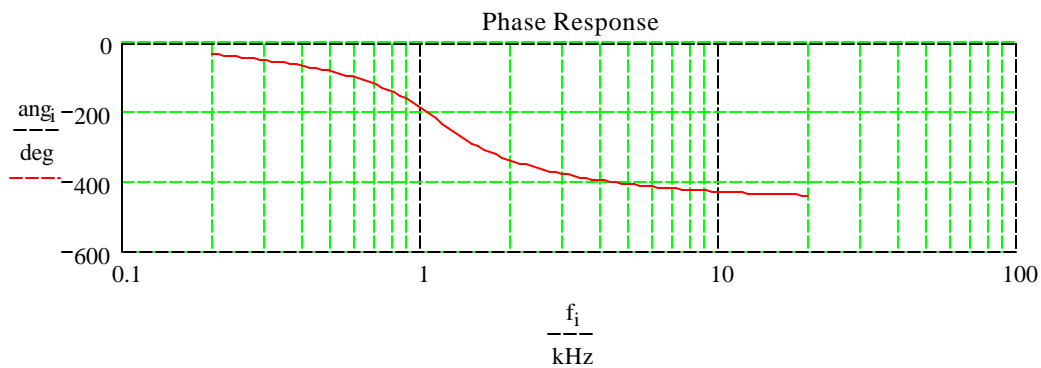
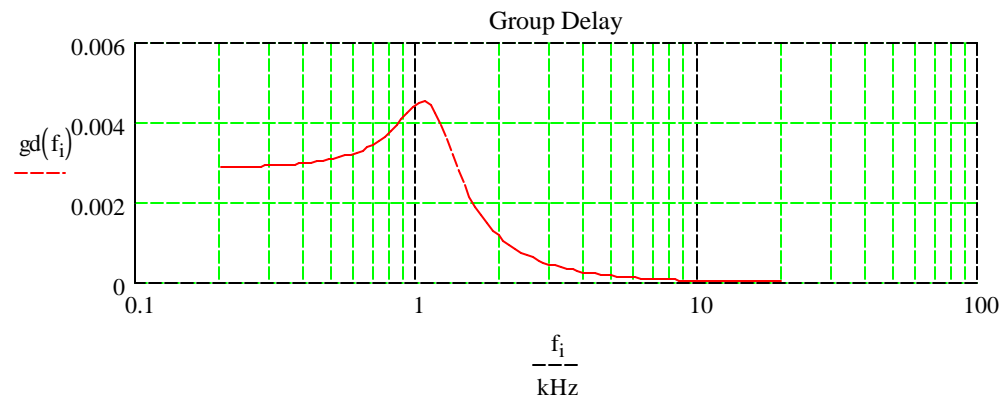
Number of Points for plotting
Frequency Index Vector

Starting Frequency for Plotting

Ending Frequency for Plotting

Frequency Vector

Phase Response



▾ Images

Element Values for Different Load and Source Impedances[1]

This method assumes the inductor is the first element

▾ Create Circuit

$$Z_S = 10\text{k}\Omega$$

$$Z_L = 20\text{k}\Omega$$

Element Values for Different Load and Source Impedances[1]

This method assumes the capacitor is the first element

▾ Create Circuit

$$Z_S = 10\text{k}\Omega$$

$$Z_L = 20\text{k}\Omega$$

Element Values for Equal Source and Load Impedances

▾ Create Circuit

$$Z_S = 10\text{k}\Omega$$

$$Z_L = 10\text{k}\Omega$$

Butterworth Coefficients

$$n := 2$$

$$\text{if} \left[\text{ceil} \left(\frac{n}{2} \right) > \frac{n}{2}, (ss + 1) \cdot \prod_{k=1}^{\frac{n-1}{2}} \left(ss^2 + 2 \cdot \cos \left(\frac{k \cdot \pi}{n} \right) \cdot ss + 1 \right), \prod_{k=1}^{\frac{n}{2}} \left[ss^2 + 2 \cdot \cos \left[\frac{(2 \cdot k - 1) \cdot \pi}{2 \cdot n} \right] \cdot ss + 1 \right] \right] \rightarrow ss^2 + 2^{\frac{1}{2}} \cdot ss + 1$$

Functions

```

butter(Atten, Ripple, fp, fs, RS) :=
  n ← 1
  fc ←  $\frac{f_s}{\left(\frac{10^{\frac{Atten}{10}}}{10} - 1\right)^{\frac{1}{2 \cdot n}}}$ 
  Aatfp ←  $10 \cdot \log \left[ 1 + \left(\frac{f_p}{f_c}\right)^{2 \cdot n} \right]$ 
  Aatfs ←  $10 \cdot \log \left[ 1 + \left(\frac{f_s}{f_c}\right)^{2 \cdot n} \right]$ 
  while (Aatfs ≤ Atten) · (Aatfp > Ripple)
    n ← n + 1
    fc ←  $\frac{f_s}{\left(\frac{10^{\frac{Atten}{10}}}{10} - 1\right)^{\frac{1}{2 \cdot n}}}$ 
    Aatfp ←  $10 \cdot \log \left[ 1 + \left(\frac{f_p}{f_c}\right)^{2 \cdot n} \right]$ 
    Aatfs ←  $10 \cdot \log \left[ 1 + \left(\frac{f_s}{f_c}\right)^{2 \cdot n} \right]$ 
  for k ∈ 1..n
    gk ←  $\sin\left(\frac{2 \cdot k - 1}{2 \cdot n} \cdot \pi\right)$ 
    numL ← if  $\left(\text{ceil}\left(\frac{n}{2}\right) > \frac{n}{2}, \frac{n-1}{2}, \frac{n}{2}\right)$ 
    numC ← if  $\left(\text{ceil}\left(\frac{n}{2}\right) > \frac{n}{2}, \frac{n+1}{2}, \frac{n}{2}\right)$ 
    for iL ∈ 1..numL
      LiL ←  $\frac{R_S}{2 \cdot \pi \cdot f_c} \cdot g_{2 \cdot iL}$ 
    for iC ∈ 1..numC
      CiC ←  $\frac{1}{2 \cdot \pi \cdot f_c \cdot R_S} \cdot g_{2 \cdot iC - 1}$ 
   $\left( \begin{array}{c} L \\ H \\ C \\ F \end{array} \right)$ 

```

Example

$x := \text{butter}(\text{Atten}, \text{Ripple}, f_p, f_s, R_s)$

$L := x_1 \cdot H$

$L^T = (0.812 \ 0.812) H$

$C := x_2 \cdot F$

$C^T = (3.103 \ 10.042 \ 3.103) nF$

Butterworth LC Filter Coefficients

$n := 2$ order of filter

$$\prod_{k=1}^{\frac{n}{2}} \left[s s^2 + 2 \cdot \cos \left[\frac{(2 \cdot k - 1) \cdot \pi}{2 \cdot n} \right] \cdot s s + 1 \right] \rightarrow s s^2 + 2^{\frac{1}{2}} \cdot s s + 1$$

n is even

S Parameter Frequency Response

S-parameter is an abbreviation for scattering parameters. S-parameters are a measure of the power gain of a network. The term scattering comes from the concept of a cue ball scattering other balls as it transfers power to them. S_{ij} is the measure of power gain from port j to port i . Specifically, the square root of power gain. For example, S_{21} , the most useful S-parameter, is the measure of the ratio of output power to the available input power. This is an important sentence. This sentence is used to perform hand calculations. The output power is easy to explain, it is V_{Orms}^2/R_L . The available input power is trickier to explain. Available input power is the maximum power that can be delivered from a source. For a source impedance of R_S , and a source voltage of V_{Srms} ; the available input power is $(V_{Srms}/2)^2/R_S$. Why, because maximum power is delivered is to a resistance of R_S . Thus a voltage division of two for the voltage, when delivering maximum available power. Thus the power gain of a network, S_{21} , is found by dividing the two powers $2 \cdot V_o/V_s \cdot \sqrt{R_S/R_L}$.

Here S-parameters for the network are calculated by finding the ABCD matrix for each element of the network, then multiplying all the matrices to get an ABCD matrix for the system.

$Z_0 := 50\Omega$

ORIGIN = 1

$$\begin{aligned}
 \text{ABCD}(\omega) := & \left. \begin{array}{l}
 \text{ABCD}_{\text{prev}} \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 \text{for } m \in 1..(\text{length}(\text{LC}) - 1) \\
 \quad \left. \begin{array}{l}
 \text{Zx} \leftarrow \frac{1}{j \cdot \omega \cdot \text{LC}_m \cdot F \cdot \Omega} \text{ if } \text{LCtype}_m = \text{"C"} \\
 \text{Zx} \leftarrow j \cdot \omega \cdot \text{LC}_m \cdot \frac{H}{\Omega} \text{ if } \text{LCtype}_m = \text{"L"} \\
 \text{Zx} \leftarrow \text{LC}_m \text{ if } \text{LCtype}_m = \text{"R"} \\
 \text{ABCDx} \leftarrow \begin{pmatrix} 1 & \text{Zx} \\ 0 & 1 \end{pmatrix} \text{ if } (\text{sp}_m = \text{"s"}) \\
 \text{ABCDx} \leftarrow \begin{pmatrix} 1 & 0 \\ \frac{1}{\text{Zx}} & 1 \end{pmatrix} \text{ if } (\text{sp}_m = \text{"p"}) \\
 \text{ABCD} \leftarrow \text{ABCDx} \cdot \text{ABCD}_{\text{prev}} \\
 \text{ABCD}_{\text{prev}} \leftarrow \text{ABCD}
 \end{array} \right\} \\
 \text{ABCD}
 \end{array}
 \right.
 \end{aligned}$$

ABCD to S parameter conversion with a common impedance on all ports using this expression from "Microwave Engineering" by Pozar.

$$\text{ABCD2S}(\text{ABCD}, Z_0) := \left[\begin{array}{l} A \leftarrow \text{ABCD}_{1,1} \\ B \leftarrow \text{ABCD}_{1,2} \\ C \leftarrow \text{ABCD}_{2,1} \\ D \leftarrow \text{ABCD}_{2,2} \end{array} \right] \left[\begin{array}{l} \left[\begin{array}{cc} 1 & \left[\begin{array}{cc} A + \frac{B \cdot \Omega}{Z_0} - C \cdot \frac{Z_0}{\Omega} - D & 2 \cdot (A \cdot D - B \cdot C) \\ \left(A + \frac{B \cdot \Omega}{Z_0} + C \cdot \frac{Z_0}{\Omega} + D \right) & 2 \end{array} \right] \\ \left(\begin{array}{cc} 1 & 10^8 \\ 0 & 1 \end{array} \right) \end{array} \right] \text{if} \left(A + \frac{B \cdot \Omega}{Z_0} + C \cdot \frac{Z_0}{\Omega} + D \right) = 0 \\ \left[\begin{array}{cc} A + \frac{B \cdot \Omega}{Z_0} - C \cdot \frac{Z_0}{\Omega} - D & 2 \cdot (A \cdot D - B \cdot C) \\ -A + \frac{B \cdot \Omega}{Z_0} - C \cdot \frac{Z_0}{\Omega} + D & \end{array} \right] \text{if} \left(A + \frac{B \cdot \Omega}{Z_0} \right)
 \end{array} \right]$$

This scattering parameter conversion routine (to convert to arbitrary source and load impedances) is given "Applied RF Techniques I" lecture notes, but is incorrect. A correct version of the routine is found on page 31 of "Microwave Amplifiers and Oscillators," by Christian Gentili.

$$\text{S}_{\text{conv}}(S, Z_{\text{Sbegin}}, Z_{\text{Send}}, Z_{\text{Lbegin}}, Z_{\text{Lend}}) := \left[\begin{array}{l} \Gamma_S \leftarrow \frac{Z_{\text{Send}} - Z_{\text{Sbegin}}}{Z_{\text{Send}} + Z_{\text{Sbegin}}} \\ \Gamma_L \leftarrow \frac{Z_{\text{Lend}} - Z_{\text{Lbegin}}}{Z_{\text{Lend}} + Z_{\text{Lbegin}}} \\ D \leftarrow (1 - \Gamma_S \cdot S_{1,1}) \cdot (1 - \Gamma_L \cdot S_{2,2}) - \Gamma_S \cdot \Gamma_L \cdot S_{1,2} \cdot S_{2,1} \\ A_1 \leftarrow \frac{1 - \overline{\Gamma_S}}{|1 - \Gamma_S|} \cdot \sqrt{1 - (|\Gamma_S|)^2} \\ A_2 \leftarrow \frac{1 - \overline{\Gamma_L}}{|1 - \Gamma_L|} \cdot \sqrt{1 - (|\Gamma_L|)^2} \\ \left[\begin{array}{cc} \frac{\overline{A_1}}{A_1} \cdot \frac{(1 - \Gamma_L \cdot S_{2,2}) \cdot (S_{1,1} - \overline{\Gamma_S}) + \Gamma_L \cdot S_{1,2} \cdot S_{2,1}}{D} & \frac{\overline{A_2}}{A_1} \cdot S_{1,2} \cdot \left[\frac{1 - \Gamma_S}{1 - \Gamma_S} \right] \\ \frac{\overline{A_1}}{A_2} \cdot S_{2,1} \cdot \frac{1 - (|\Gamma_L|)^2}{D} & \frac{\overline{A_2}}{A_2} \cdot \frac{(1 - \Gamma_S \cdot S_{1,1}) \cdot (S_{2,2} - \overline{\Gamma_L}) + \Gamma_S \cdot S_{1,2} \cdot S_{2,1}}{D} \end{array} \right]
 \end{array} \right]$$

$$S_{50}(\omega) := \text{ABCD2S}(\text{ABCD}(\omega), Z_0)$$

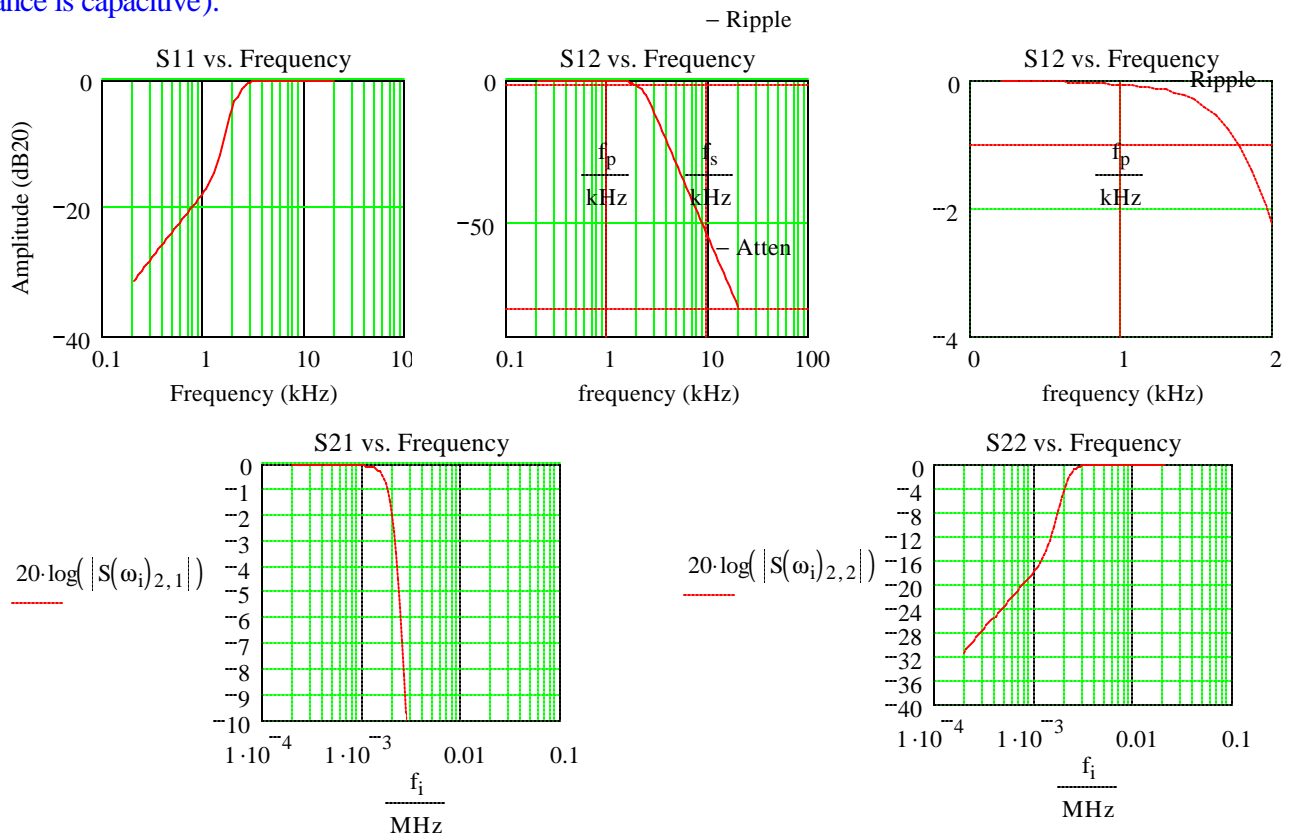
S-parameters of ideal matching network with actual load and source impedances.

$$Z_L := R_S \quad Z_S := R_S$$

$$S(\omega) := \text{S}_{\text{conv}}(S_{50}(\omega), Z_0, Z_S, Z_0, Z_L)$$

50 Ohm S-parameters of ideal matching network.

Plots of lossy and ideal S-parameters of the matching network versus frequency. Be careful when using these plots, as they do not reflect the change in source and load impedance vs. frequency (for example if driver output impedance is capacitive).



References

- [1] *Microwave Electronic Circuit Technology*, by Yoshihiro Konishi, Marcel Dekker Publishing, New York, 1998, Filter Design Equations: pp 199-
 2] *Passive and Active Filters, Theory and Implementations*, by Wai-Kai Chen, John Wiley & Sons, New York, 1986, pp. 177-184

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