Butterworth LC Filter Designer

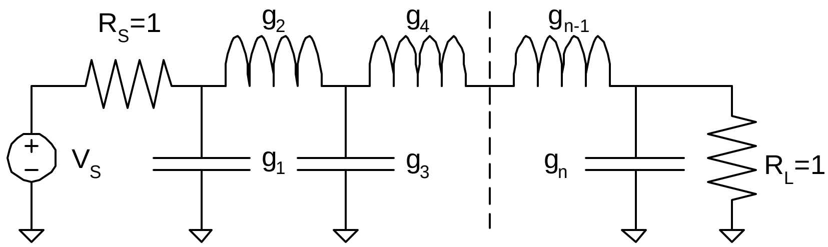

Fig. 1: LC filter used for odd-order analysis

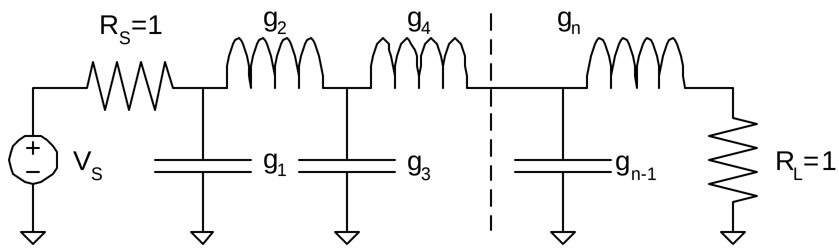

Fig. 2: Filter used for even order analysis

▸ useful functions and identities
▸ Units
▸ Constants

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Introduction

This routine calculates the required inductances and capacitances for a lowpass LC butterworth filter. It is special in that it also sizes the components to deliver maximum power from a given source resistance to a different load resistance. Reactive portions of the source and load must be remove subtracted the filter network, so it is usually desirable to choose an odd order network for maximum power transfer. Given the lowpass LC coefficients, the network can easily be transformed into a bandpass, highpass, notch filter based on other topologies: gm/C, active RC, etc. A butterworth filter has the following power transfer function, where the order of the filter is given by n:

$$A = 10 \cdot \log\left[1 + \left(\frac{f}{f_c}\right)^{2 \cdot n}\right]$$

Inputs

$Atten := 80dB$	**Stop-Band Attenuation**
$Ripple := 1dB$	**Maximum Passband Ripple**
$f_p := 1kHz$	**Passband corner frequency**
$f_s := 10kHz$	**Stopband corner frequency**
$R_S := 10k\Omega$	**Source Impedance**
$R_L := 20k\Omega$	**Load Impedance**

Order and -3dB Corner Frequency Estimation

$$n = \mathrm{ceil}\left(\frac{\ln\left(10^{\frac{Atten}{10}} - 1\right)}{2\cdot\ln\left(\frac{f_s}{f_c}\right)}\right)$$

Required Order of Butterworth Filter for 3dB droop

Function 1: Holds Atten Constant and chooses n to meet Ripple

$$nf_c := \left|\begin{array}{l} n \leftarrow 1 \\ f_c \leftarrow \dfrac{f_s}{\left(10^{\frac{Atten}{10}} - 1\right)^{\frac{1}{2\cdot n}}} \\ Aatfp \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_p}{f_c}\right)^{2\cdot n}\right] \\ Aatfs \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_s}{f_c}\right)^{2\cdot n}\right] \\ \text{while } (Aatfs \le Atten)\cdot(Aatfp > Ripple) \\ \quad \left|\begin{array}{l} n \leftarrow n + 1 \\ f_c \leftarrow \dfrac{f_s}{\left(10^{\frac{Atten}{10}} - 1\right)^{\frac{1}{2\cdot n}}} \\ Aatfp \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_p}{f_c}\right)^{2\cdot n}\right] \\ Aatfs \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_s}{f_c}\right)^{2\cdot n}\right] \end{array}\right. \\ \begin{pmatrix} n \\ \dfrac{f_c}{Hz} \end{pmatrix} \end{array}\right.$$

$n := nf_{c_1}$ $\quad n = 5$ Number of Poles

$f_c := nf_{c_2}\cdot Hz$ $\quad f_c = 1.585\,kHz$ 3 dB Corner Frequency

$$Aatfp := 10\cdot\log\left[1 + \left(\frac{f_p}{f_c}\right)^{2\cdot n}\right] \qquad Aatfp = 0.043\,dB$$ Passband Ripple

$$Aatfs := 10\cdot\log\left[1 + \left(\frac{f_s}{f_c}\right)^{2\cdot n}\right] \qquad Aatfs = 80\,dB$$ Stopband Attenuation

Function 2: Holds Ripple Constant and chooses n to meet Atten

$$nf_c := \left|\begin{array}{l} n \leftarrow 1 \\ f_c \leftarrow \dfrac{f_p}{\left(10^{\frac{Ripple}{10}} - 1\right)^{\frac{1}{2\cdot n}}} \\ Aatfp \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_p}{f_c}\right)^{2\cdot n}\right] \\ Aatfs \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_s}{f_c}\right)^{2\cdot n}\right] \\ \text{while } (Aatfs \le Atten)\cdot(Aatfp \ge Ripple) \\ \quad \left|\begin{array}{l} n \leftarrow n + 1 \\ f_c \leftarrow \dfrac{f_p}{\left(10^{\frac{Ripple}{10}} - 1\right)^{\frac{1}{2\cdot n}}} \\ Aatfp \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_p}{f_c}\right)^{2\cdot n}\right] \\ Aatfs \leftarrow 10\cdot\log\left[1 + \left(\dfrac{f_s}{f_c}\right)^{2\cdot n}\right] \end{array}\right. \\ \begin{pmatrix} n \\ \dfrac{f_c}{Hz} \end{pmatrix} \end{array}\right.$$

$n := nf_{c_1}$ $\qquad n = 5$ $\qquad$ Number of Poles

$f_c := nf_{c_2}\cdot Hz$ $\qquad f_c = 1.145\,kHz$ $\qquad$ 3 dB Corner Frequency

$Aatfp := 10\cdot\log\left[1 + \left(\dfrac{f_p}{f_c}\right)^{2\cdot n}\right]$ $\qquad Aatfp = 1\,dB$ $\qquad$ Passband Ripple

$Aatfs := 10\cdot\log\left[1 + \left(\dfrac{f_s}{f_c}\right)^{2\cdot n}\right]$ $\qquad Aatfs = 94.132\,dB$ $\qquad$ Stopband Attenuation

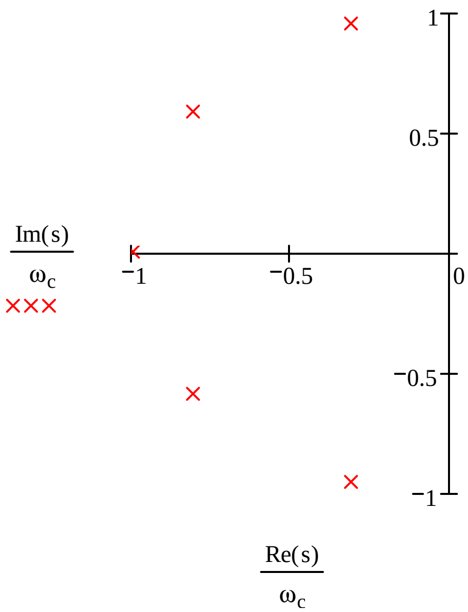
Poles and Zeros

$k := 1 .. n$ Index Vector for poles

$\omega_c := 2 \cdot \pi \cdot f_c$

$s_k := \omega_c \cdot e^{\frac{j \cdot (2 \cdot k + n - 1) \cdot \pi}{2 \cdot n}}$ Poles

$s^T = (-2.223 + 6.84i \quad -5.819 + 4.227i \quad -7.192 \quad -5.819 - 4.227i \quad -2.223 - 6.84i)\ \text{kHz}$



Quadratic Sections

$$\frac{1}{\frac{s^2}{re^2+im^2}+\left(\frac{2\cdot re}{re^2+im^2}\right)\cdot s+1}=\frac{1}{\left(\frac{s}{\omega}\right)^2+\frac{1}{Q}\cdot\frac{s}{\omega}+1}=\frac{1}{\left[\left(\frac{s}{\omega}\right)^2+2\cdot\zeta\cdot\frac{s}{\omega}+1\right]}$$

$$m := 1..\,\mathrm{floor}\left(\frac{n}{2}\right)$$

$$\omega_m := \sqrt{\mathrm{Re}\left(s_m\right)^2+\mathrm{Im}\left(s_m\right)^2}$$

$$\frac{\omega^T}{2\cdot\pi} = (1.145 \quad 1.145)\,\mathrm{kHz}$$

Quadratic center frequenc butterworth)

$$Q_m := \frac{-\omega_m}{2\cdot\mathrm{Re}\left(s_m\right)}$$

$$Q^T = (1.618 \quad 0.618)$$

Quadratic Q

$$\zeta_m := \frac{-\mathrm{Re}\left(s_m\right)}{\omega_m}$$

$$\zeta^T = (0.309 \quad 0.809)$$

Quadratic Damping Fact

$$\omega_{\mathrm{ceil}\left(\frac{n}{2}\right)} := \mathrm{if}\left(\mathrm{ceil}\left(\frac{n}{2}\right)>\frac{n}{2},-s_{\mathrm{ceil}\left(\frac{n}{2}\right)},0\,\frac{\mathrm{rad}}{\mathrm{sec}}\right)$$

$$\frac{\omega^T}{2\cdot\pi} = (1.145 \quad 1.145 \quad 1.145)\,\mathrm{kHz}$$

First Order Section

Transfer Function

$$\alpha_k := \prod_{m=1}^{k} \frac{\cos\left[\frac{(m-1)\cdot\pi}{2\cdot n}\right]}{\sin\left(\frac{m\cdot\pi}{2\cdot n}\right)} \qquad \alpha^T = (3.236 \quad 5.236 \quad 5.236 \quad 3.236 \quad 1)$$

Transfer Function Coefficients (α_n always equals 1)

$$M(s) := \frac{1}{1 + \sum_{k=1}^{n} \alpha_k \cdot \left(\frac{s}{\omega_c}\right)^k}$$

Transfer Function

$$gd(f) := \frac{d}{df} arg\left[\frac{1}{1 + \sum_{k=1}^{n} \alpha_k \cdot \left(\frac{j \cdot f}{f_c}\right)^k}\right]$$

Group Delay Function

$$
\begin{aligned}
phase(p) := \; & p \leftarrow \overrightarrow{arg(p)} \\
& \text{return } p \text{ if } last(p) < 2 \\
& wrap \leftarrow 0 \\
& \text{for } i \in 2..last(p) \\
& \quad wrap \leftarrow wrap - 2\cdot\pi \text{ if } \left(p_i - p_{i-1} + wrap\right) > 3 \\
& \quad wrap \leftarrow wrap + 2\cdot\pi \text{ if } \left(p_i - p_{i-1} + wrap\right) < -3 \\
& \quad p_i \leftarrow p_i + wrap \\
& p
\end{aligned}
$$

Phase Function

$$M_2(f) := \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2\cdot n}}}$$

Magnitude Transfer Function for 3dB droop

$$\left(\left|S_{21}(s)\right|\right)^2 = \frac{K_n}{1 + \varepsilon^2 \cdot \left(\frac{f}{f_c}\right)^{2\cdot n}} \qquad \alpha_{max} = 10\cdot\log\left(1 + \varepsilon^2\right) \qquad \varepsilon = \sqrt{10^{\frac{\alpha_{max}}{10}} - 1}$$

Power Transfer Function for αmax dB droop

$$M_3(s) := \frac{1}{\sqrt{1 + (-1)^n \cdot \left(\frac{s}{\omega_c}\right)^{2\cdot n}}}$$

Transfer Function (Eq. 2.10b)

Plotting

$num := 100$ — Number of Points for plotting

$i := 1 .. num$ — Frequency Index Vector

$f_{start} := \frac{f_p}{5}$ — Starting Frequency for Plotting

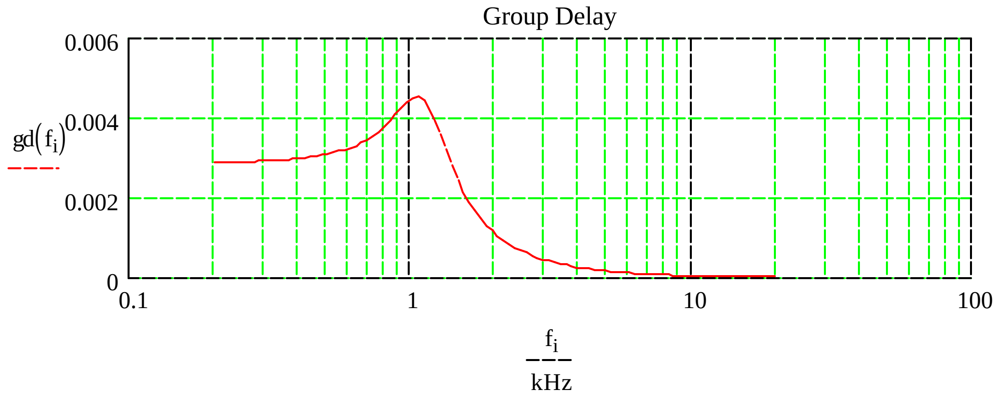
$f_{stop} := f_s \cdot 2$ — Ending Frequency for Plotting

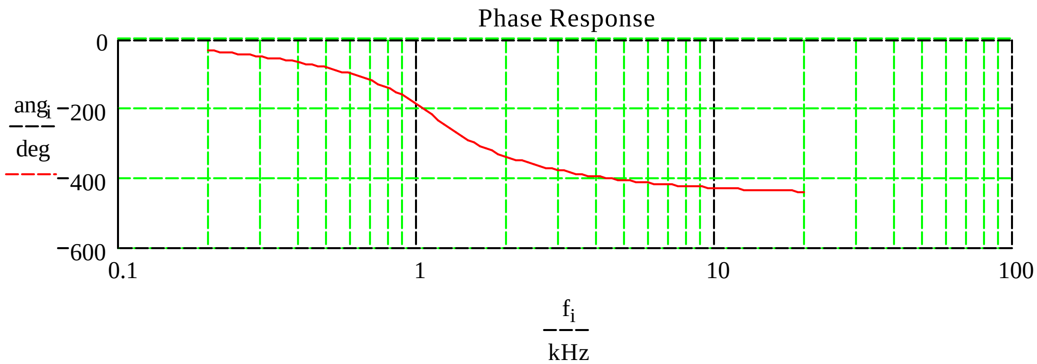
$f_i := \left(\frac{f_{stop}}{f_{start}}\right)^{\frac{i-1}{num-1}} \cdot f_{start}$ — Frequency Vector

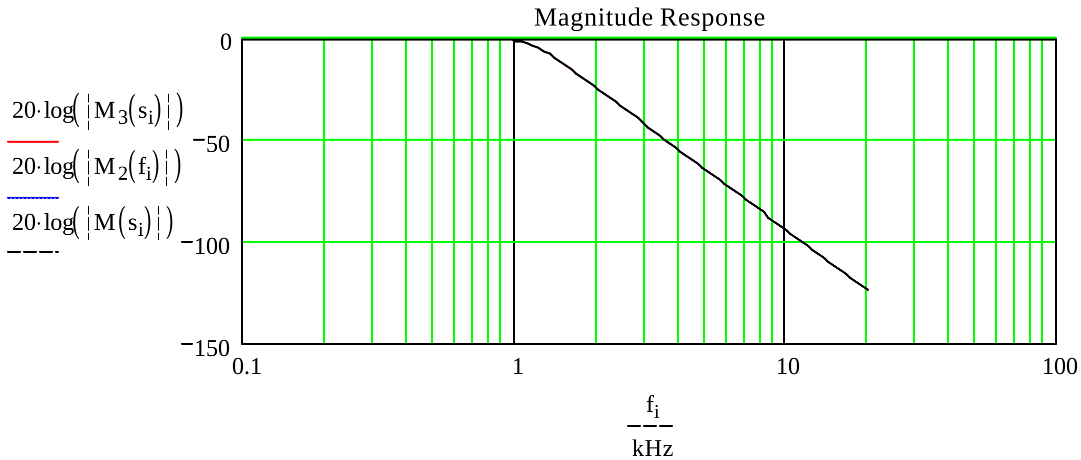
$\omega_i := 2 \cdot \pi \cdot f_i$

$s_i := j \cdot \omega_i$

$ang := phase\left(\overrightarrow{M(s)}\right)$ — Phase Response







▶ Images

Element Values for Different Load and Source Impedances[1]

This method assumes the inductor is the first element

▶ Create Circuit

$Z_S = 10\,k\Omega$

$Z_L = 20\,k\Omega$

Element Values for Different Load and Source Impedances[1]

This method assumes the capacitor is the first element

▶ Create Circuit

$Z_S = 10\,k\Omega$

$Z_L = 20\,k\Omega$

Element Values for Equal Source and Load Impedances

▶ Create Circuit

$Z_S = 10\,k\Omega$

$Z_L = 10\,k\Omega$

Butterworth Coefficients

$n := 2$

$$\text{if}\left[\text{ceil}\left(\frac{n}{2}\right) > \frac{n}{2}, (ss+1)\cdot\prod_{k=1}^{\frac{n-1}{2}}\left(ss^2 + 2\cdot\cos\left(\frac{k\cdot\pi}{n}\right)\cdot ss + 1\right), \prod_{k=1}^{\frac{n}{2}}\left[ss^2 + 2\cdot\cos\left[\frac{(2\cdot k-1)\cdot\pi}{2\cdot n}\right]\cdot ss + 1\right]\right] \rightarrow ss^2 + 2^{\frac{1}{2}}\cdot ss + 1$$

Functions

$$\mathrm{butter}(Atten, Ripple, f_p, f_s, R_S) :=$$

$$n \leftarrow 1$$

$$f_c \leftarrow \frac{f_s}{\left(10^{\frac{Atten}{10}} - 1\right)^{\frac{1}{2 \cdot n}}}$$

$$Aatfp \leftarrow 10 \cdot \log\left[1 + \left(\frac{f_p}{f_c}\right)^{2 \cdot n}\right]$$

$$Aatfs \leftarrow 10 \cdot \log\left[1 + \left(\frac{f_s}{f_c}\right)^{2 \cdot n}\right]$$

while $(Aatfs \le Atten) \cdot (Aatfp > Ripple)$

$$n \leftarrow n + 1$$

$$f_c \leftarrow \frac{f_s}{\left(10^{\frac{Atten}{10}} - 1\right)^{\frac{1}{2 \cdot n}}}$$

$$Aatfp \leftarrow 10 \cdot \log\left[1 + \left(\frac{f_p}{f_c}\right)^{2 \cdot n}\right]$$

$$Aatfs \leftarrow 10 \cdot \log\left[1 + \left(\frac{f_s}{f_c}\right)^{2 \cdot n}\right]$$

for $k \in 1 .. n$

$$g_k \leftarrow \sin\left(\frac{2 \cdot k - 1}{2 \cdot n} \cdot \pi\right)$$

$$numL \leftarrow \mathrm{if}\left(\mathrm{ceil}\left(\frac{n}{2}\right) > \frac{n}{2}, \frac{n-1}{2}, \frac{n}{2}\right)$$

$$numC \leftarrow \mathrm{if}\left(\mathrm{ceil}\left(\frac{n}{2}\right) > \frac{n}{2}, \frac{n+1}{2}, \frac{n}{2}\right)$$

for $iL \in 1 .. numL$

$$L_{iL} \leftarrow \frac{R_S}{2 \cdot \pi \cdot f_c} \cdot g_{2 \cdot iL}$$

for $iC \in 1 .. numC$

$$C_{iC} \leftarrow \frac{1}{2 \cdot \pi \cdot f_c \cdot R_S} \cdot g_{2 \cdot iC - 1}$$

$$\begin{pmatrix} \frac{L}{H} \\ \frac{C}{F} \end{pmatrix}$$

Example

$x := \text{butter}(\text{Atten}, \text{Ripple}, f_p, f_s, R_S)$

$L := x_1 \cdot H$ $\quad L^T = (0.812 \quad 0.812)\, H$

$C := x_2 \cdot F$ $\quad C^T = (3.103 \quad 10.042 \quad 3.103)\, nF$

Butterworth LC Filter Coefficients

$n := 2$ order of filter

$$\prod_{k=1}^{\frac{n}{2}} \left[ss^2 + 2 \cdot \cos\left[\frac{(2 \cdot k - 1) \cdot \pi}{2 \cdot n}\right] \cdot ss + 1 \right] \rightarrow ss^2 + 2^{\frac{1}{2}} \cdot ss + 1$$

n is even

S Parameter Frequency Response

S-parameter is an abbreviation for scattering parameters. S-parameters are a measure of the power gain of a network. The term scattering comes from the concept of a cue ball scattering other balls as it transfers power to them. S_{ij} is the measure of power gain from port j to port i. Specifically, the square root of power gain. For example, S_{21}, the most useful S-parameter, is the measure of the ratio of output power to the available input power. This is an important sentence. This sentence is used to perform hand calculations. The output power is easy to explain, it is $V_{Orms}{}^2/R_L$. The available input power is trickier to explain. Available input power is the maximum power that can be delivered from a source. For a source impedance of R_S, and a source voltage of V_{Srms}; the available input power is $(V_{Srms}/2)^2/R_S$. Why, because maximum power is delivered is to a resistance of R_S. Thus a voltage division of two for the voltage, when delivering maximum available power. Thus the power gain of a network, S_{21}, is found by dividing the two powers 2*V_o/V_S*sqrt(R_S/R_L).

Here S-parameters for the network are calculated by finding the ABCD matrix for each element of the network, then multiplying all the matrices to get an ABCD matrix for the system.

$Z_0 := 50\Omega$ ORIGIN = 1

$$\mathrm{ABCD}(\omega) := \left|\begin{array}{l} \mathrm{ABCD_{prev}} \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{for } m \in 1..(\mathrm{length}(LC)-1) \\ \quad \left|\begin{array}{l} Zx \leftarrow \dfrac{1}{j\cdot\omega\cdot LC_m\cdot F\cdot\Omega} \;\text{ if }\; LCtype_m = \text{"C"} \\ Zx \leftarrow j\cdot\omega\cdot LC_m\cdot\dfrac{H}{\Omega} \;\text{ if }\; LCtype_m = \text{"L"} \\ Zx \leftarrow LC_m \;\text{ if }\; LCtype_m = \text{"R"} \\ ABCDx \leftarrow \begin{pmatrix} 1 & Zx \\ 0 & 1 \end{pmatrix} \;\text{ if }\; (sp_m = \text{"s"}) \\ ABCDx \leftarrow \begin{pmatrix} 1 & 0 \\ \dfrac{1}{Zx} & 1 \end{pmatrix} \;\text{ if }\; (sp_m = \text{"p"}) \\ ABCD \leftarrow ABCDx\cdot ABCD_{prev} \\ ABCD_{prev} \leftarrow ABCD \end{array}\right. \\ \mathrm{ABCD} \end{array}\right.$$

ABCD to S parameter conversion with a common impedance on all ports using this expression from "Microwave Engineering" by Pozar.

$$\mathrm{ABCD2S}(ABCD, Z_0) := \left| \begin{array}{l} A \leftarrow ABCD_{1,1} \\ B \leftarrow ABCD_{1,2} \\ C \leftarrow ABCD_{2,1} \\ D \leftarrow ABCD_{2,2} \\ \left[\frac{1}{\left(A + \frac{B\cdot\Omega}{Z_0} + C\cdot\frac{Z_0}{\Omega} + D\right)} \cdot \begin{bmatrix} A + \frac{B\cdot\Omega}{Z_0} - C\cdot\frac{Z_0}{\Omega} - D & 2\cdot(A\cdot D - B\cdot C) \\ 2 & -A + \frac{B\cdot\Omega}{Z_0} - C\cdot\frac{Z_0}{\Omega} + D \end{bmatrix} \right] \text{ if } \left(A + \frac{B\cdot\Omega}{Z_0} \cdots \right. \\ \begin{pmatrix} 1 & 10^8 \\ 0 & 1 \end{pmatrix} \text{ if } \left(A + \frac{B\cdot\Omega}{Z_0} + C\cdot\frac{Z_0}{\Omega} + D\right) = 0 \end{array} \right.$$

This scattering parameter conversion routine (to convert to arbitrary source and load impedances) is given "Applied RF Techniques I" lecture notes, but is incorrect. A correct version of the routine is found on page 31 of "Microwave Amplifiers and Oscillators," by Christian Gentili.

$$S_{conv}(S, Z_{Sbegin}, Z_{Send}, Z_{Lbegin}, Z_{Lend}) := \left| \begin{array}{l} \Gamma_S \leftarrow \frac{Z_{Send} - Z_{Sbegin}}{Z_{Send} + Z_{Sbegin}} \\ \Gamma_L \leftarrow \frac{Z_{Lend} - Z_{Lbegin}}{Z_{Lend} + Z_{Lbegin}} \\ D \leftarrow \left(1 - \Gamma_S\cdot S_{1,1}\right)\cdot\left(1 - \Gamma_L\cdot S_{2,2}\right) - \Gamma_S\cdot\Gamma_L\cdot S_{1,2}\cdot S_{2,1} \\ A_1 \leftarrow \frac{1 - \overline{\Gamma_S}}{\left|1 - \Gamma_S\right|}\cdot\sqrt{1 - \left(\left|\Gamma_S\right|\right)^2} \\ A_2 \leftarrow \frac{1 - \overline{\Gamma_L}}{\left|1 - \Gamma_L\right|}\cdot\sqrt{1 - \left(\left|\Gamma_L\right|\right)^2} \\ \begin{bmatrix} \frac{\overline{A_1}}{A_1}\cdot\frac{\left(1 - \Gamma_L\cdot S_{2,2}\right)\cdot\left(S_{1,1} - \overline{\Gamma_S}\right) + \Gamma_L\cdot S_{1,2}\cdot S_{2,1}}{D} & \frac{\overline{A_2}}{A_1}\cdot\frac{S_{1,2}\cdot\left[1 - \cdots\right.}{\cdots} \\ \frac{\overline{A_1}}{A_2}\cdot S_{2,1}\cdot\frac{1 - \left(\left|\Gamma_L\right|\right)^2}{D} & \frac{\overline{A_2}}{A_2}\cdot\frac{\left(1 - \Gamma_S\cdot S_{1,1}\right)\cdot\left(S_{2,2} \cdots\right.}{\cdots} \end{bmatrix} \end{array} \right.$$

$$S_{50}(\omega) := \mathrm{ABCD2S}(ABCD(\omega), Z_0)$$

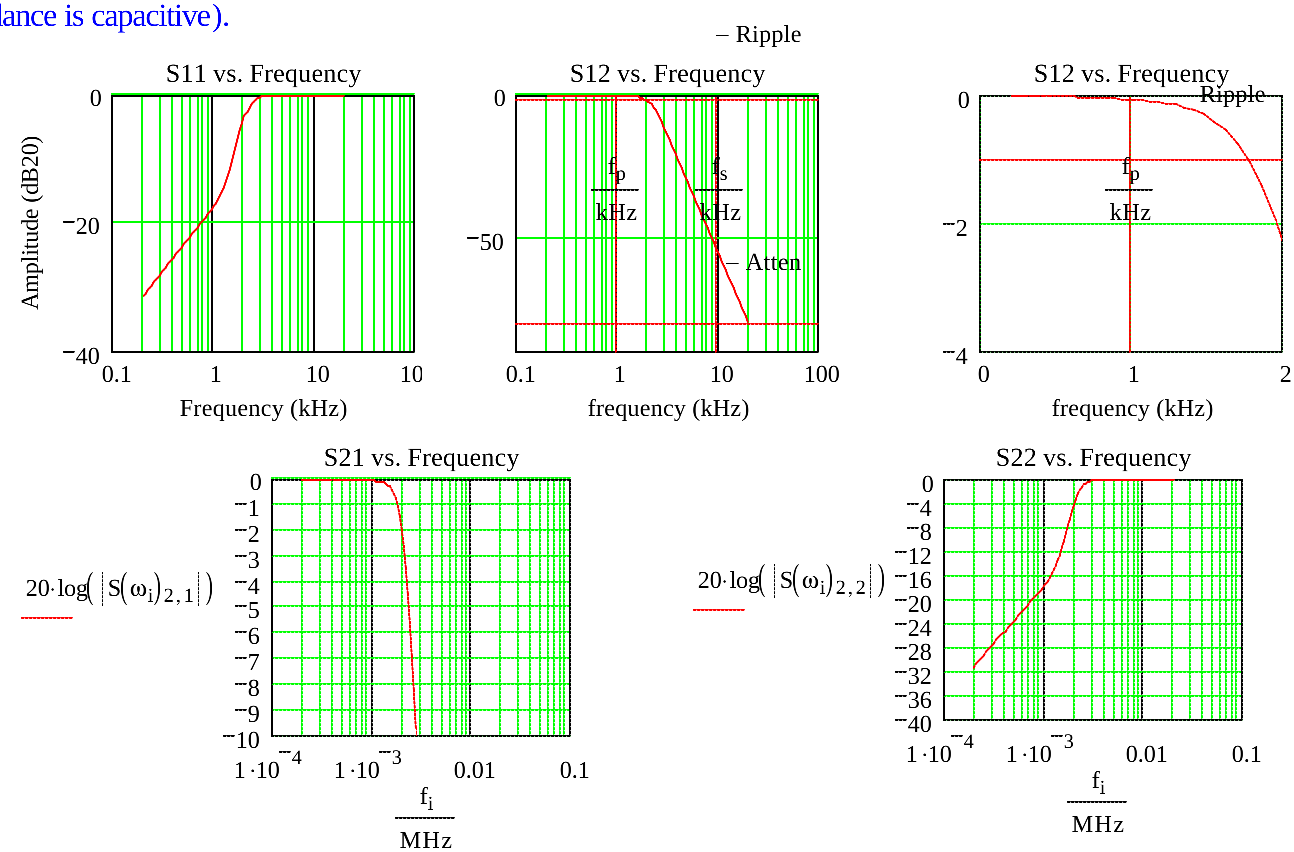
S-parameters of ideal matching network with actual load and source impedances.

$Z_L := R_S$ $\qquad Z_S := R_S$

$$S(\omega) := S_{conv}(S_{50}(\omega), Z_0, Z_S, Z_0, Z_L)$$

50 Ohm S-parameters of ideal matching network.

Plots of lossy and ideal S-paramaters of the matching network verses frequency. Be careful when using these plots, as they do not reflect the change in source and load impedance vs. frequency (for example if driver output impedance is capacitive).



References

[1]Microwave Electronic Circuit Technology, by Yoshihiro Konishi, Marcel Dekker Publishing, New York, 1998, Filter Design Equations: pp 199-

2]Passive and Active Filters, Theory and Implementations, by Wai-Kai Chen, John Wiley & Sons, New York, 1986, pp. 177-184

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