Optimal Phase Margin for Minimal Settling Time

$$A(s) = \frac{\omega_u}{s} \cdot \frac{1}{1 + \frac{s}{p2}}$$

▶ useful functions and identities
▶ Units
▶ Constants

Table of Contents

Introduction

What is the truth to the rumor that the optimal phase margin for minimal settling time is 45 deg, 60 deg, 64 deg, or critical damping? What phase margin corresponds to critical damping? These questions will be answered in this report with specific application to second order systems consisting of a dominant and non-dominant poles.

Inputs

SNR := 80dB

Phase Margin, Q and Damping Factor

The open loop response of most second order systems can be represented in the following form. This form assumes infinite gain at DC. This assumption results in neglible error for DC gains greater than 10, which is the case for most operational amplifiers with DC gains typically greater than 500.

$$A_{ol} = \frac{\omega_u}{s} \cdot \frac{1}{1 + \frac{s}{p2}}$$

The closed loop transfer function given this forward path gain is given by the following equations. In this closed loop response the feedback factor is assumed to unity, as it will be lumped into ω_u.

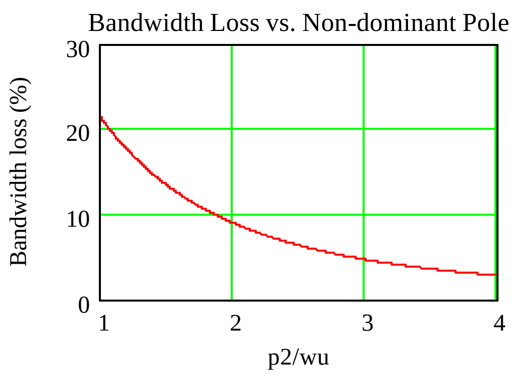
$$A_{cl} = \frac{A_{ol}}{1 + A_{ol}} = \frac{1}{\frac{s^2}{p2 \cdot \omega_u} + \frac{s}{\omega_u} + 1} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + \frac{1}{Q} \cdot \frac{s}{\omega_0} + 1} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2 \cdot \zeta \cdot \frac{s}{\omega_0} + 1}$$

The closed loop gain can be expressed in terms of Q, damping factor, phase margin, or as a function of non-dominant pole location with respect to unity gain bandwidth.

$$\omega_0 = \omega_u \cdot \sqrt{p2\_\omega u}$$

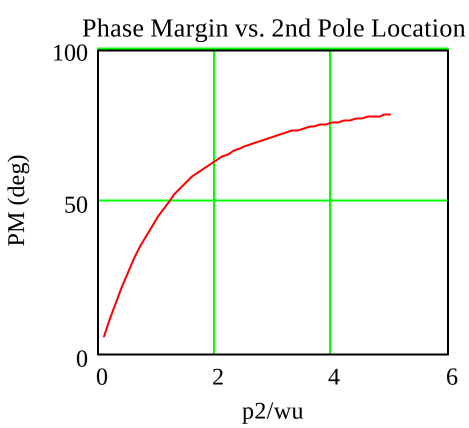
$$Q(p2\_\omega u) := \frac{1}{\sqrt{p2\_\omega u}} \qquad \zeta(p2\_\omega u) := \frac{\sqrt{p2\_\omega u}}{2} \qquad PM(p2\_\omega u) := 90 - \frac{180}{\pi} \cdot \operatorname{atan}\left(\frac{1}{p2\_\omega u}\right)$$

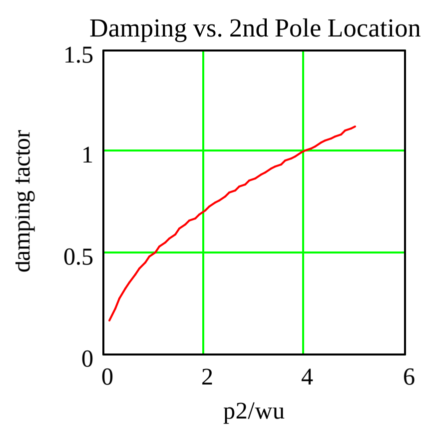
This phase margin expression assumes, the unity gain bandwidth is entirely determined by the dominant pole. In practice, the non-dominant pole will reduce the real unity gain bandwidth from the expected value. For properly placed non-dominant pole this error is only about 5%, as shown in the following figure:

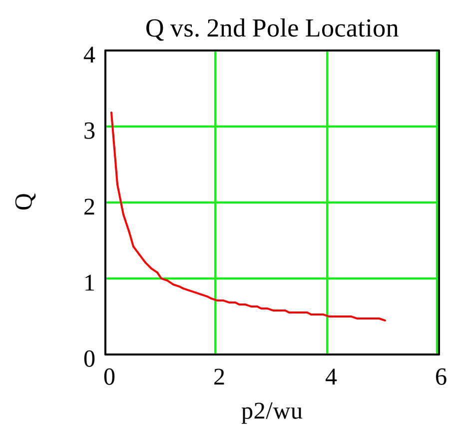


As a designer we have control over the pole locations, which is swept in the following plots to show the relationship between non-dominant pole location and Q, phase margin and damping factor.

$$num := 50 \qquad i := 1 .. num \qquad \omega 0t_i := \frac{i - 1}{num - 1} \cdot 20 \qquad p2\_\omega u_i := \frac{i}{num} \cdot 5$$







Transient Response

In this section, the settling time is found. This first requires the transient response be found. For a second order system, three possible equations can result, depending on whether the response is under, over, or critically damped.

Critically damped

For a critcally damped system, the damping factor is one. This response has a transient response and settling error given by the following expression:

$$h(\omega0t) := 1 - (\omega0t + 1)\cdot \exp(-\omega0t)$$

$$err0(\omega0t) := -(\omega0t + 1)\cdot \exp(-\omega0t)$$

Underdamped

For damping factors less than one, the system is underdamped. This is the preferred operating condition for almost all systems for minimum settling time.

$$p_1(p2\_\omega u) := \zeta(p2\_\omega u) + \sqrt{\zeta(p2\_\omega u)^2 - 1} \qquad p_2(p2\_\omega u) := \zeta(p2\_\omega u) - \sqrt{\zeta(p2\_\omega u)^2 - 1}$$

$$A_{cl} = \frac{1}{\left(\frac{s}{p_1} + 1\right)\cdot\left(\frac{s}{p_2} + 1\right)}$$

$$h(t) = 1 + e^{(-\zeta\cdot\omega_0\cdot t)}\cdot\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\cdot \sin\left(\sqrt{1-\zeta^2}\cdot\omega_0\cdot t\right) - \cos\left(\sqrt{1-\zeta^2}\cdot\omega_0\cdot t\right)\right)$$

$$err(t) = |h(t) - 1| = e^{(-\zeta\cdot\omega_0\cdot t)}\cdot\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\cdot \sin\left(\sqrt{1-\zeta^2}\cdot\omega_0\cdot t\right) - \cos\left(\sqrt{1-\zeta^2}\cdot\omega_0\cdot t\right)\right)$$

$$err1(p2\_\omega u, \omega0t) := e^{\left(-\frac{\sqrt{p2\_\omega u}}{2}\cdot\omega0t\right)}\cdot\left(\frac{1}{\sqrt{\frac{4}{p2\_\omega u} - 1}}\cdot \sin\left(\sqrt{1 - \frac{p2\_\omega u}{4}}\cdot\omega0t\right) - \cos\left(\sqrt{1 - \frac{p2\_\omega u}{4}}\cdot\omega0t\right)\right)$$

Overdamped

An overdamped system is very slow. However it is often used in systems with large process variations, to avoid crossing into extreme underdamped scenarios.

$$h(t) = 1 + \frac{p_2}{(p_1 - p_2)}\cdot \exp(-p_1\cdot\omega_0\cdot t) - \frac{p_1}{(p_1 - p_2)}\cdot \exp(-p_2\cdot\omega_0\cdot t)$$

$$err2(p2\_\omega u, \omega0t) := \frac{p_2(p2\_\omega u)}{\sqrt{p2\_\omega u - 4}}\cdot \exp(-p_1(p2\_\omega u)\cdot\omega0t) - \frac{p_1(p2\_\omega u)}{\sqrt{p2\_\omega u - 4}}\cdot \exp(-p_2(p2\_\omega u)\cdot\omega0t)$$

Overall Error vs. Non-Dominant Pole Location

The three transient responses are merged into the following expression:

$$err(p2\_\omega u, \omega 0t) := if\left(p2\_\omega u = 4, err0(\omega 0t), if\left(\frac{\sqrt{p2\_\omega u}}{2} \geq 1, err2(p2\_\omega u, \omega 0t), err1(p2\_\omega u, \omega 0t)\right)\right)$$

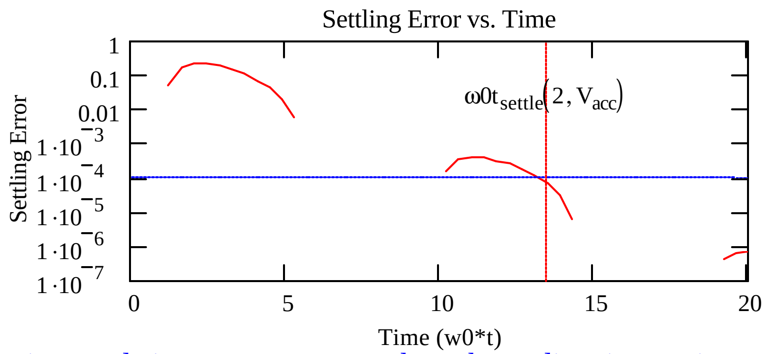
The required settling accuracy is calculated from the SNR requirements as follows

$$V_{acc} := 10^{\frac{-SNR}{20}}$$

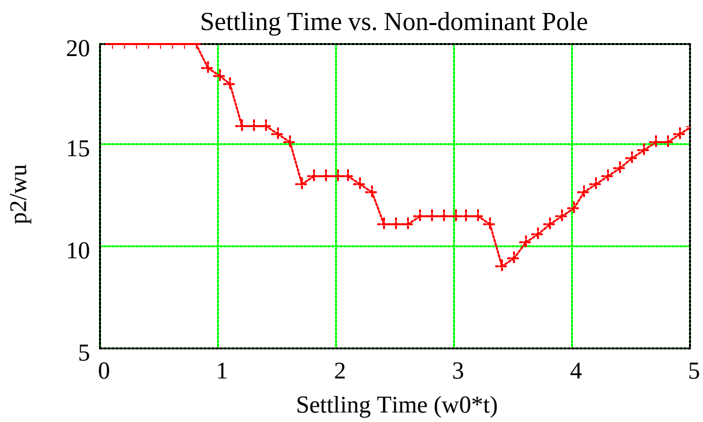
From the transient response we can find the settling time

$$\begin{aligned} \omega 0t_{settle}(p2\_\omega u, V_{acc}) := \; & t_{val} \leftarrow 0 \\ & \text{for } i \in 2..num \\ & \quad t_{val} \leftarrow if\left[\left(err(p2\_\omega u, \omega 0t_i) \leq V_{acc}\right)\cdot\left(\left|err(p2\_\omega u, \omega 0t_{i-1})\right| > V_{acc}\right), \omega 0t_i, t_{val}\right] \\ & t_{val} \end{aligned}$$

Settling time is best found by plotting settling error on a logarithmic axis verses time.



When the nondominant pole is swept, we can see how the settling time varies



Now we can find the optimum pole placement for settling given a required settling accuracy:

$$\begin{aligned} p2\_\omega u_{opt}(V_{acc}) := \; & \omega 0t_{setmin} \leftarrow 1000000 \\ & opt \leftarrow 0 \\ & \text{for } i \in 2..num \\ & \quad \omega 0t_{set} \leftarrow \omega 0t_{settle}(p2\_\omega u_i, V_{acc}) \\ & \quad opt \leftarrow if(\omega 0t_{set} < \omega 0t_{setmin}, p2\_\omega u_i, opt) \\ & \quad \omega 0t_{setmin} \leftarrow if(\omega 0t_{set} < \omega 0t_{setmin}, \omega 0t_{set}, \omega 0t_{setmin}) \\ & opt \end{aligned}$$

$$p2\_\omega u_{opt}(V_{acc}) = 3.4$$

This corresponds to a phase margin of

$$PM_{opt}(V_{acc}) := 90 - \frac{180}{\pi} \cdot \operatorname{atan}\left(\frac{1}{p2\_\omega u_{opt}(V_{acc})}\right) \qquad PM_{opt}(V_{acc}) = 73.61$$

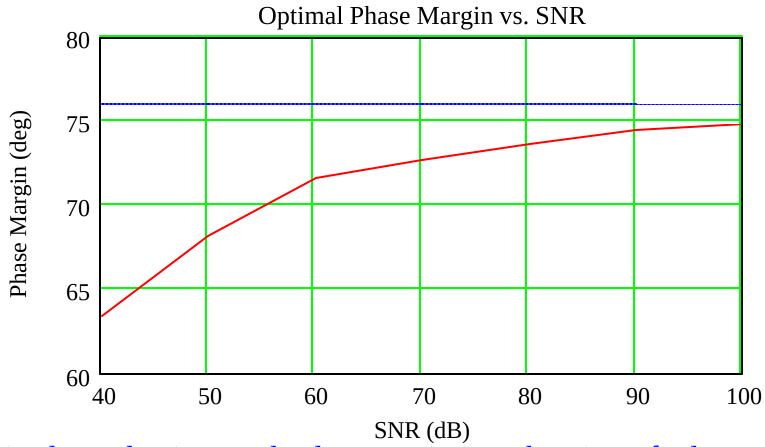
num := 7

i := 1.. num

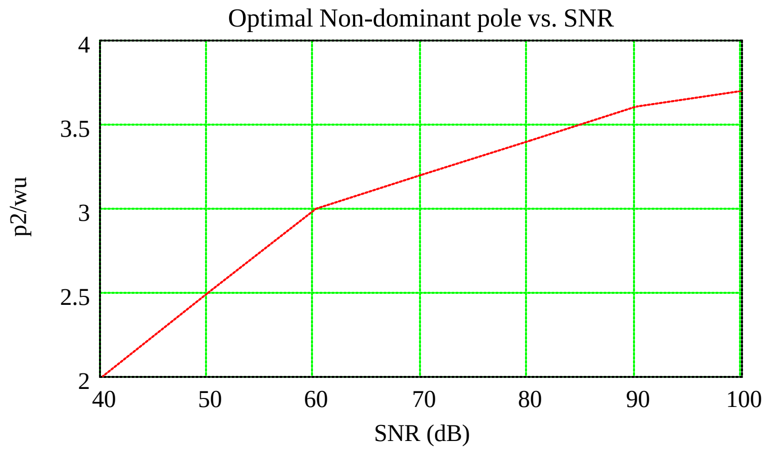
$SNR_i := 30 + i \cdot 10$

$V_{acc_i} := 10^{\frac{-SNR_i}{20}}$

Now for the most useful plot of this report: Optimal phase margin verses SNR requirements. 76 degrees represents critical damping. The optimal phase margin approaches this as the SNR requirements get large.

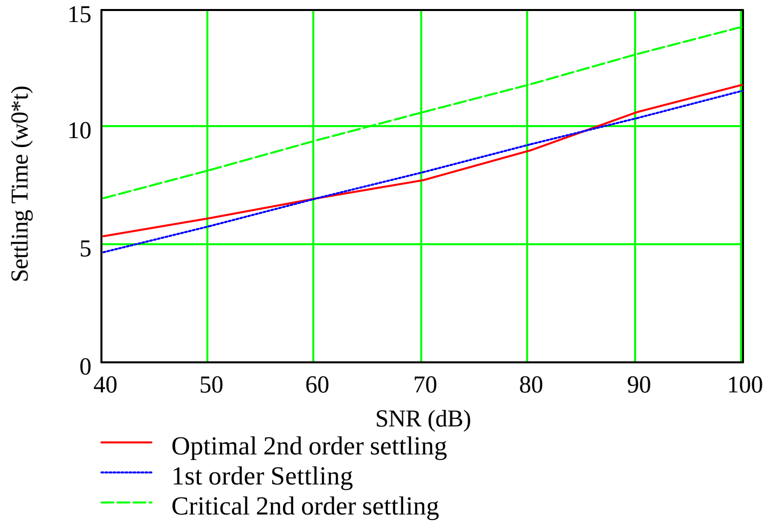


Similarly, the optimal non-dominant pole placement approaches $4 * \omega_u$ for large SNR requirements



It is interesting to that optimized settling time is the same as first order settling with a bandwidth of ω_0, and in most cases 40% faster than critical settling.

$$\omega0t_{setlin}(SNR) := \frac{SNR}{20} \cdot \ln(10)$$



Extra comments:
-Third order loops are trickier to analyze, because their are two variables involved in the settling: phase margin and gain margin.
-This analysis assumes linearity is limited by the exponential settling error.
-With ideal 1st order linear settling, incomplete settling does not affect SDR.
-For continuous time filters and PLLs the optimum phase margin is low, in the 50 degree range.

Copyright and Trademark Notice